Final Exam   M340

For each of the following decide if it is true or false and give a brief explanation for your answer.

1. The initial value problem $y'(t) = t^2 + \sqrt{y(t)}$, $y(1) = 0$ has a unique local solution.

FALSE
$f(t,y) = t^2 + \sqrt{y(t)}$ is continuous at $t = 1, y = 0$ so a solution exists
$\partial_y f(t,y) = \frac{1}{2\sqrt{y}}$ is undefined at $t = 1, y = 0$ so the solution is not unique

2. The autonomous equation $y'(t) = 2y(y+2)(y-3)$ has 2 stable equilibrium points.

FALSE
$f(y) = 2y(y+2)(y-3)$ is zero at $y = -2, 0, 3$
$f'(y) = 6y^2 - 4y - 12$ is positive at $y = -2, 3$ and is negative at $y = 0$
then $y = -2, 3$ are unstable and $y = 0$ is stable

3. The equation

$$2x''(t) + 4x'(t) + 34x(t) = 0, \quad x(0) = 0, x'(0) = -2$$

describes the motion of a spring-mass system starting from the equilibrium position with an upward initial velocity and oscillating with constant amplitude for all $t > 0$.

FALSE
the auxiliary equation, $2r^2 + 4r + 34 = 2[r^2 + 2r + 1 + 16] = 0$
has roots $x = -1 \pm 4i$ so the linearly independent solutions are
$e^{-t}\cos 4t$ and $e^{-t}\sin 4t$ so the mass oscillates with decreasing amplitude.

4. If $L[y(t)] = 0$ is a homogeneous, linear, ordinary differential equation with constant coefficients, then $L[e^{rt}] = P(r)e^{rt}$. Then $y(t) = e^{rt}$ solves the ODE if r is a root of the auxiliary equation $P(r) = 0$.

TRUE
Since $L[e^{rt}] = P(r)e^{rt}$, and e^{rt} is never zero, $L[e^{rt}] = 0$ if and only if $P(r) = 0$.

5. The matrix

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

has an eigenvalue with algebraic multiplicity 3 and geometric multiplicity 2.

TRUE
A is triangular so $\lambda = 3, 3, 3$ has algebraic multiplicity 3.

since

$$A - 3I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

has rank equal to 1, the dimension of $N[A - 3I] = n - r = 3 - 1 = 2$,
so the geometric multiplicity is 2.

6. If A is the matrix from problem 5, then $e^{tA} = e^{3t}\begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

TRUE
evidently, $(A - 3I)^2 = 0$ so,
$e^{tA} = e^{3t}e^{(A-3I)t} = e^{3t}[I + t(A - 3I) + t^2/2(A - 3I)^2 + \cdots]$

$$= e^{3t}[I + t(A - 3I) + 0] = e^{3t}\begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solve each of the following problems:

7. Find the general solution to:

$$t^2y'(t) + 2ty(t) = 5t^4.$$

Then find the unique solution that satisfies $y(2) = 12.$

note that the derivative of t^2 is $2t$ so $t^2y'(t) + 2ty(t) = \frac{d}{dt}(t^2y(t))$
Then

$$\frac{d}{dt}(t^2y(t)) = 5t^4$$

$$t^2y(t) = t^5 + C$$

$$y_{Gen}(t) = t^3 + Ct^{-2}$$

This is the general solution.
The IC implies $y(2) = 8 + C/4 = 12$ so $C = 16$ gives the unique solution to the IVP

8. The matrix

$$A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$$

has an eigenvalue, $\lambda = 2 - 3i,$ with the corresponding eigenvector, $\vec{E} = \begin{bmatrix} i \\ 1 \end{bmatrix}$. Find the real valued general solution for $\vec{X}'(t) = A\vec{X}(t)$.

$\vec{X}(t) = e^{(2-3i)t}\,\vec{E}$ is a complex valued solution to $\vec{X}'(t) = A\vec{X}(t)$
Then

$$\begin{aligned}
\vec{X}(t) &= e^{2t}e^{i3t}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix} + i\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} \\
&= e^{2t}[\cos 3t + i\sin 3t]\left[\vec{U} + i\vec{V}\right] \\
&= e^{2t}\left\{\cos 3t\vec{U} - \sin 3t\vec{V}\right\} + ie^{2t}\left\{\cos 3t\vec{V} - \sin 3t\vec{U}\right\} \\
&= \vec{Z}_1(t) + i\vec{Z}_2(t)
\end{aligned}$$

and $\vec{Z}_1(t)$ and $\vec{Z}_2(t)$ are linearly independent real valued solutions to $\vec{X}'(t) = A\vec{X}(t)$.
A real valued general solution for $\vec{X}'(t) = A\vec{X}(t)$ is then $C_1\vec{Z}_1(t) + C_2\vec{Z}_2(t)$

9. Find the general solution for $\vec{X}'(t) = A\vec{X}$ if A is the matrix from problem 5.

$(A - 3I)\vec{X} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\vec{X}$ *so* $x_2 = 0$ and x_1, x_3 are free.

Then $\vec{E}_1 = [1,0,0]^T$ and $\vec{E}_2 = [0,0,1]^T$ are independent e-vectors for $\lambda = 3$.
Let

$$(A - 3I)\vec{W} = \vec{E}_1$$

$$\text{so} \quad (A - 3I)^2\vec{W} = (A - 3I)\vec{E}_1 = \vec{0}$$

Then $e^{At}\vec{W}$ solves $\vec{X}'(t) = A\vec{X}$ and

$$\begin{aligned}
e^{At}\vec{W} &= e^{3t}e^{(A-3I)t}\vec{W} \\
&= e^{3t}\left[\vec{W} + t(A - 3I)\vec{W} + 0\right] \\
&= e^{3t}\left[\vec{W} + t\vec{E}_1\right]
\end{aligned}$$

Then the general solution to the system can be written as

$$\vec{X}(t) = C_1e^{3t}\vec{E}_1 + C_2e^{3t}\vec{E}_2 + C_3e^{3t}\left[\vec{W} + t\vec{E}_1\right]$$

10. Use undetermined coefficients to find the general solution for: $y'(t) + 2y(t) = 8t.$ Then find the unique solution that satisfied $y(0) = 2$.

The homogeneous solution is $y(t) = Ce^{-2t}$
a particular solution will have the form $y_p = At + B$
substituting this into the ODE gives

$$A + 2(At + B) = 8t$$

then $2A = 8$ and $A + 2B = 0$ so $A = 4$ and $B = -2$.
The general solution of the ODE is then $y(t) = Ce^{-2t} + 4t - 2$

Since $y(0) = C - 2 = 2,$ we get $C = 4$ and $y(t) = 4e^{-2t} + 4t - 2$
is the unique solution to the IVP

11. Find the general solution for: $y''(t) + 4y'(t) + 20y(t) = 0.$

The auxiliary equation here is $r^2 + 4r + 20 = r^2 + 4r + 4 + 16 = 0$
Then $r = -2 \pm 4i$
The general solution of the ODE is $y(t) = e^{-2t}[C_1 \cos 4t + C_2 \sin 4t]$

12. Use the Laplace transform to solve:

$$y''(t) - y'(t) - 2y(t) = t^3 \quad y(0) = 1, \quad y'(0) = -2$$

Transforming the equation

$$\begin{aligned} s^2\hat{y}(s) - s + 2 - (s\hat{y}(s) - 1) - 2\hat{y}(s) &= \hat{f}(s) \\ (s^2 - s - 2)\hat{y}(s) &= s - 3 + \hat{f}(s) \\ \hat{y}(s) &= \frac{s-3}{(s-2)(s+1)} + \frac{\hat{f}(s)}{(s-2)(s+1)} \end{aligned}$$

Then

$$\frac{s-3}{(s-2)(s+1)} = \frac{4}{3(s+1)} - \frac{1}{3(s-2)}$$

so

$$\hat{y}(s) = \frac{4}{3(s+1)} - \frac{1}{3(s-2)} + \left[\frac{1}{3(s-2)} - \frac{1}{3(s+1)}\right]\hat{f}(s)$$

and using the table to invert the transform,

$$y(t) = \frac{4}{3}e^{-t} - \frac{1}{3}e^{2t} + \frac{1}{3}\int_0^t [e^{2(t-\tau)} - e^{-(t-\tau)}]\tau^3 d\tau$$